

Assignment 10

Coverage: 16.4 in Text.

Optional reading: Wiki “Green’s theorem”, “cross product”.

Exercises: 16.4 no 7, 11, 14, 23, 26, 27, 30, 37, 39.

Hand in 16.4 no 14, 27, 39 by April 13.

Supplementary Problems

1. A vector field \mathbf{F} is called radial if $\mathbf{F}(x, y, z) = f(r)(x, y, z)$, $r = \sqrt{x^2 + y^2 + z^2}$, for some function f . Show that every radial vector field is conservative. You may assume it is C^1 in \mathbb{R}^3 .
2. Let $F = (P, Q)$ be a C^1 -vector field in \mathbb{R}^2 away from the origin. Suppose that $P_y = Q_x$. Show that for any simple closed curve C enclosing the origin and oriented in positive direction, one has

$$\oint_C P dx + Q dy = \lim_{\varepsilon \rightarrow 0} \varepsilon \int_0^{2\pi} [-P(\varepsilon \cos \theta, \varepsilon \sin \theta) \sin \theta + Q(\varepsilon \cos \theta, \varepsilon \sin \theta) \cos \theta] d\theta .$$

What happens when C does not enclose the origin?

3. (Optional) We identify the complex plane with \mathbb{R}^2 by $x + iy \mapsto (x, y)$. A complex-valued function f has its real and imaginary parts respectively given by $u(x, y) = \operatorname{Re} f(z)$ and $v(x, y) = \operatorname{Im} f(z)$. Note that u and v are real-valued functions. The function f is called differentiable at z if

$$\frac{df}{dz}(z) = \lim_{w \rightarrow 0} \frac{f(z+w) - f(z)}{w} ,$$

exists.

- (a) Show that f is differentiable at z implies that the partial derivatives of u and v exist and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, hold. Hint: Take $w = h, ih$, where $h \in \mathbb{R}$ and then let $h \rightarrow 0$.
- (b) Propose a definition of $\int_C f dz$, where C is an oriented curve in the plane, in terms of the line integrals involving u and v .
- (c) Suppose that f is differentiable everywhere in \mathbb{C} . Show that for every simple closed curve C ,

$$\oint_C f dz = 0 .$$

Exercises 16.4

Circulation and Flux

In Exercises 5–14, use Green's Theorem to find the counterclockwise circulation and outward flux for the field \mathbf{F} and curve C .

14. $\mathbf{F} = \left(\tan^{-1} \frac{y}{x} \right) \mathbf{i} + \ln(x^2 + y^2) \mathbf{j}$

C : The boundary of the region defined by the polar coordinate inequalities $1 \leq r \leq 2, 0 \leq \theta \leq \pi$

Using Green's Theorem

Calculating Area with Green's Theorem If a simple closed curve C in the plane and the region R it encloses satisfy the hypotheses of Green's Theorem, the area of R is given by

Green's Theorem Area Formula

$$\text{Area of } R = \frac{1}{2} \oint_C x \, dy - y \, dx$$

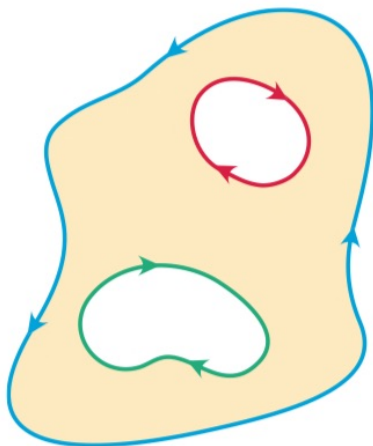
The reason is that by Equation (4), run backward,

$$\begin{aligned} \text{Area of } R &= \iint_R dy \, dx = \iint_R \left(\frac{1}{2} + \frac{1}{2} \right) dy \, dx \\ &= \oint_C \frac{1}{2} x \, dy - \frac{1}{2} y \, dx. \end{aligned}$$

Use the Green's Theorem area formula given above to find the areas of the regions enclosed by the curves in Exercises 25–28.

27. The astroid $\mathbf{r}(t) = (\cos^3 t) \mathbf{i} + (\sin^3 t) \mathbf{j}, \quad 0 \leq t \leq 2\pi$

39. Regions with many holes Green's Theorem holds for a region R with any finite number of holes as long as the bounding curves are smooth, simple, and closed and we integrate over each component of the boundary in the direction that keeps R on our immediate left as we go along (see accompanying figure).



- a. Let $f(x, y) = \ln(x^2 + y^2)$ and let C be the circle $x^2 + y^2 = a^2$. Evaluate the flux integral

$$\oint_C \nabla f \cdot \mathbf{n} \, ds.$$

- b. Let K be an arbitrary smooth, simple closed curve in the plane that does not pass through $(0, 0)$. Use Green's Theorem to show that

$$\oint_K \nabla f \cdot \mathbf{n} \, ds$$

has two possible values, depending on whether $(0, 0)$ lies inside K or outside K .